

# Hochschild Cohomology of Cubic Surfaces

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## Abstract

We consider the polynomial algebra  $\mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, z_2, z_3]$  and the polynomial  $f := z_1^3 + z_2^3 + z_3^3 + 3qz_1z_2z_3$ , where  $q \in \mathbb{C}$ . Our aim is to compute the Hochschild homology and cohomology of the cubic surface

$$\mathcal{X}_f := \{\mathbf{z} \in \mathbb{C}^3 / f(\mathbf{z}) = 0\}.$$

For explicit computations, we shall make use of a method suggested by M. Kontsevich. Then, we shall develop it in order to determine the Hochschild homology and cohomology by means of multivariate division and Groebner bases. Some formal computations with Maple are also used.

**Keywords:** Hochschild cohomology; Hochschild homology; cubic surfaces; Groebner bases; algebraic resolution; quantization; star-products.

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## 1 Introduction and main results

### 1.1 Deformation quantization

Let us consider a mechanical system  $(M, \mathcal{F}(M))$ , where  $M$  is a Poisson manifold and  $\mathcal{F}(M)$  the algebra of regular functions on  $M$ . In classical mechanics, we study the (commutative) algebra  $\mathcal{F}(M)$ , that is to say the observables of the classical system. The evolution of an observable  $f$  is determined by the Hamiltonian equation  $\dot{f} = \{f, H\}$ , where  $H \in \mathcal{F}(M)$  is the Hamiltonian of the system. But quantum mechanics, where the physical system is described by a (non commutative) algebra of operators on a Hilbert space, gives more precise results than its classical analogue. Therefore it is important to be able to quantize it.

One available method is geometric quantization, which allows us to construct in an explicit way a Hilbert space and an algebra  $\mathcal{C}$  of operators on this space (see the book [6]), and to associate to every classical observable  $f \in \mathcal{F}(M)$  a quantum one  $\hat{f} \in \mathcal{C}$ . This very interesting method presents the drawback of being seldom applicable.

That is why other methods, such as asymptotic quantization and deformation quantization, have been introduced. The latter, described in 1978 by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer in [3], is a good alternative: instead of constructing an algebra of operators on a Hilbert space, we define a formal deformation of  $\mathcal{F}(M)$ . This is given by the algebra of formal power series  $\mathcal{F}(M)[[\hbar]]$ , endowed with some associative, but not always commutative, star-product:

$$f * g = \sum_{j=0}^{\infty} m_j(f, g) \hbar^j \tag{1}$$

where the maps  $m_j$  are bilinear and where  $m_0(f, g) = fg$ . Then quantization is given by the map  $f \mapsto \hat{f}$ , where the operator  $\hat{f}$  satisfies  $\hat{f}(g) = f * g$ .

We can wonder in which cases a Poisson manifold admits such a quantization. The answer was given by Kontsevich in [11] who constructed a star-product on every Poisson manifold. Besides, he proved

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that if  $M$  is a smooth manifold, then the equivalence classes of formal deformations of the zero Poisson bracket are in bijection with equivalence classes of star-products. Moreover, as a consequence of the Hochschild-Kostant-Rosenberg theorem, every abelian star-product is equivalent to a trivial one.

In the case where  $M$  is a singular algebraic variety, say

$$M = \{\mathbf{z} \in \mathbb{C}^n / f(\mathbf{z}) = 0\},$$

where  $f$  belongs to  $\mathbb{C}[\mathbf{z}]$ , we shall consider the algebra of functions on  $M$ , i. e. the quotient algebra  $\mathbb{C}[\mathbf{z}] / \langle f \rangle$ . So the above mentioned result is not always valid. However, the deformations of the algebra  $\mathcal{F}(M)$ , defined by the formula (1), are always classified by the Hochschild cohomology of  $\mathcal{F}(M)$ , and we are led to the study of the Hochschild cohomology of  $\mathbb{C}[\mathbf{z}] / \langle f \rangle$ .

## 1.2 Some recent related works

Several recent articles were devoted to the study of particular cases, for Hochschild as well as for Poisson homology and cohomology:

C. Roger and P. Vanhaecke, in [16], calculate the Poisson cohomology of the affine plane  $\mathbb{C}^2$ , endowed with the Poisson bracket  $f \partial_1 \wedge \partial_2$ , where  $f$  is a homogeneous polynomial. They express it in terms of the number of irreducible components of the singular locus  $\{\mathbf{z} \in \mathbb{C}^2 / f(\mathbf{z}) = 0\}$ .

M. Van den Bergh and A. Pichereau, in [18], [13] and [14], are interested in the case where  $f$  is a weighted homogeneous polynomial with an isolated singularity at the origin. They compute the Poisson homology and cohomology, which may be expressed in particular in terms of the Milnor number of the space  $\mathbb{C}[z_1, z_2, z_3] / \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle$ .

In [1], Jacques Alev, Marco A. Farinati, Thierry Lambre and Andrea L. Solotar establish a fundamental result: they compute all the Hochschild homology and cohomology spaces of  $A_n(\mathbb{C})^G$ , where  $A_n(\mathbb{C})$  is the Weyl algebra, for every finite subgroup  $G$  of  $\mathbf{Sp}_{2n}\mathbb{C}$ . It is an interesting and classical question to compare the Hochschild homology and cohomology of  $A_n(\mathbb{C})^G$  with the Poisson homology and cohomology of the ring of invariants  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^G$ , which is a quotient algebra of the form  $\mathbb{C}[\mathbf{z}] / \langle f_1, \dots, f_m \rangle$ .

Finally, C. Fronsdaal studies in [4] Hochschild homology of singular curves of the plane. Besides, the appendix of this article gives another way to calculate the Hochschild cohomology in the more general case of complete intersections.

## 1.3 Main results of the paper

Let us consider the polynomial algebra  $\mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, z_2, z_3]$  and the polynomial

$$f := z_1^3 + z_2^3 + z_3^3 + 3qz_1z_2z_3 \in \mathbb{C}[\mathbf{z}],$$

where  $q \in \mathbb{C}$ . The surface defined by  $f$ , i.e.  $\mathcal{X}_f := \{\mathbf{z} \in \mathbb{C}^3 / f(\mathbf{z}) = 0\}$ , is called a cubic surface. The aim of this article is to compute the Hochschild homology and cohomology of this algebraic variety. Besides, we will consider the desingularisation of  $\mathcal{X}_f$ , and also compute its Hochschild homology and cohomology. Let us denote by  $A$  the quotient algebra  $A := \mathbb{C}[\mathbf{z}] / \langle f \rangle$ .

The main result of the article is given by two theorems:

### Theorem 1

Let  $HH^p$  be the space of Hochschild cohomology of degree  $p$ . Then we have

$$HH^0 = A, \quad HH^1 = \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \oplus \mathbb{C}^8, \quad HH^2 = \{g \nabla f / g \in A\} \oplus \mathbb{C}^8, \quad \text{and } \forall p \geq 3, \quad HH^p = \mathbb{C}^8.$$

### Theorem 2

Let  $HH_p$  be the space of Hochschild homology of degree  $p$ . Then we have

$$HH_0 = A, \quad HH_1 = \nabla f \wedge A^3, \quad HH_2 = A^3 / (\nabla f \wedge A^3), \quad \text{and } \forall p \geq 3, \quad HH_p = \mathbb{C}^8.$$

For explicit computations, we shall make use of a method suggested by M. Kontsevich in the appendix of [4]. Then, we shall develop it in order to determine the Hochschild homology and cohomology by means of multivariate division and Groebner bases. Some formal computations with the software Maple will be used all along our study.

## 1.4 Outline of the paper

In Section 2 we state the main theorems about Groebner bases and regular sequences which are useful for our computations. Then we recall the classical definitions about Hochschild homology and cohomology, and the important results about deformations of associative algebras.

Section 3 is the main section of the article: we compute the Hochschild cohomology of  $\mathcal{X}_f$ . We begin by giving some results about the Koszul complex, then we describe the cohomology spaces, and we compute them by means of Groebner bases and formal computations.

Finally, in Section 4, we study the case of Hochschild homology of cubic surfaces.

## 2 Framework and tools

### 2.1 Groebner bases and regular sequences

For each ideal  $J$  of  $\mathbb{C}[\mathbf{z}]$ , we denote by  $J_A$  the image of  $J$  by the canonical projection

$$\mathbb{C}[\mathbf{z}] \rightarrow A = \mathbb{C}[\mathbf{z}]/\langle f_1, \dots, f_m \rangle.$$

Similarly if  $(g_1, \dots, g_r) \in A^r$  we denote by  $\langle g_1, \dots, g_r \rangle_A$  the ideal of  $A$  generated by  $(g_1, \dots, g_r)$ .

Besides, if  $g \in \mathbb{C}[\mathbf{z}]$ , and if  $J$  is an ideal of  $\mathbb{C}[\mathbf{z}]$ , we set

$$\text{Ann}_J(g) := J : \langle g \rangle = \{h \in \mathbb{C}[\mathbf{z}] \mid hg = 0 \pmod{J}\}.$$

In particular,  $g$  does not divide 0 in  $\mathbb{C}[\mathbf{z}]/J$  if and only if  $\text{Ann}_J(g) = J$ .

Finally, we denote by  $\nabla g$  the gradient of a polynomial  $g \in \mathbb{C}[\mathbf{z}]$ , and by  $\langle \nabla g \rangle$  the ideal generated by the three partial derivatives of  $g$ .

Moreover, we use the notation  $\partial_j$  for the partial derivative with respect to  $z_j$ .

Here, we recall some important results about Groebner bases (see [15] for more details). For  $g \in \mathbb{C}[\mathbf{z}]$ , we denote by  $lt(g)$  its leading term (for the lexicographic order  $\preceq$ ). Given a sequence of polynomials  $[g_1, \dots, g_k]$ , we say that  $g$  reduced with respect to  $[g_1, \dots, g_k]$ , if  $g$  is zero or if no one of the terms of  $g$  is divisible by one of the elements  $lt(g_1), \dots, lt(g_k)$ .

Given a non trivial ideal  $J$  of  $\mathbb{C}[\mathbf{z}]$ , a *Groebner basis* of  $J$  is a finite subset  $G$  of  $J \setminus \{0\}$  such that for every  $f \in J \setminus \{0\}$ , there exists  $g \in G$  such that  $lt(g)$  divides  $lt(f)$ .

#### Proposition 3

*Let  $G$  be a Groebner basis of an ideal  $J$  of  $\mathbb{C}[\mathbf{z}]$ . Then every element of  $J$  that is reduced with respect to  $G$ , is zero.*

*For every  $f \in \mathbb{C}[\mathbf{z}]$ , there exists a unique polynomial  $r_G(f) \in \mathbb{C}[\mathbf{z}]$ , reduced with respect to  $G$ , such that  $f \equiv r_G(f) \pmod{J}$ .*

The unique polynomial  $r_G(f)$  defined above is called the normal form of  $f$ . So we have

$$\forall f \in \mathbb{C}[\mathbf{z}], f \in J \Leftrightarrow r_G(f) = 0.$$

The following theorem will be useful for the computation of the Hochschild cohomology of cubic surfaces. Given  $J$  a non trivial ideal of  $\mathbb{C}[\mathbf{z}]$ , the *set of  $J$ -standard terms*, is the set of monomials of  $\mathbb{C}[\mathbf{z}]$  except the set of dominant terms of non zero elements of  $J$ .

**Theorem 4 (Macaulay)**

The set of  $J$ -standard terms is a basis of the quotient  $\mathbb{C}$ -vector space  $\mathbb{C}[\mathbf{z}] / J$ .

Let  $A = \bigoplus_{d=0}^{\infty} A_d$  a graded  $\mathbb{C}$ -algebra of Krull dimension  $n$  (i.e. the maximal number of elements of  $A$  that are algebraically independent on  $\mathbb{C}$  is  $n$ ).

Let  $H(A_+)$  be the set of elements of  $A$  that are homogeneous of positive degree. A sequence  $[\theta_1, \dots, \theta_n]$  of elements of  $H(A_+)$  is a *homogeneous system of parameters* if  $A$  is a module of finite type on the ring  $\mathbb{C}[\theta_1, \dots, \theta_n]$ . In particular, the elements  $\theta_1, \dots, \theta_n$  are algebraically independent.

According to the normalization lemma of Noether<sup>2</sup>, a homogeneous system of parameters always exists.

Let us now introduce a technical tool: a sequence  $[\theta_1, \dots, \theta_n]$  of  $n$  elements of  $A$  is called a *regular sequence* if, for every  $j \in \llbracket 1, n \rrbracket$ ,  $\theta_j$  is not a divisor of zero of  $A / \langle \theta_1, \dots, \theta_{j-1} \rangle$ .

If the elements  $\theta_1, \dots, \theta_n$  are algebraically independent (and this is the case in particular if  $[\theta_1, \dots, \theta_n]$  is a homogeneous system of parameters), then  $[\theta_1, \dots, \theta_n]$  is a regular sequence if and only if  $A$  is a free module on the algebra  $\mathbb{C}[\theta_1, \dots, \theta_n]$ .

**Theorem 5 (see [17])**

Let  $A$  be a graded  $\mathbb{C}$ -algebra, and  $[\theta_1, \dots, \theta_n]$  a homogeneous system of parameters. The two following points are equivalent:

- $A$  is a free module of finite type on  $\mathbb{C}[\theta_1, \dots, \theta_n]$ , i.e.

$$A = \bigoplus_{i=1}^m \eta_i \mathbb{C}[\theta_1, \dots, \theta_n], \quad (2)$$

- For every homogeneous system of parameters  $[\phi_1, \dots, \phi_n]$ ,  $A$  is a free module of finite type on  $\mathbb{C}[\phi_1, \dots, \phi_n]$ .

The meaning of this theorem is the following: in a graded  $\mathbb{C}$ -algebra, if there exists a homogeneous system of parameters that is a regular sequence, then every homogeneous system of parameters is a regular sequence. In particular, if  $f \in \mathbb{C}[z_1, z_2, z_3]$  is a weight homogeneous polynomial with an isolated singularity at the origin, then  $[f, \partial_3 f, \partial_2 f]$  is a homogeneous system of parameters. Now  $[z_1, z_2, z_3]$  is a homogeneous system of parameters that is trivially a regular sequence, therefore  $[f, \partial_3 f, \partial_2 f]$  is a regular sequence.

**2.2 Hochschild homology and cohomology and deformations of algebras**

- Consider an associative  $\mathbb{C}$ -algebra, denoted by  $A$ . The Hochschild cohomological complex of  $A$  is

$$\mathbb{C}^0(A) \xrightarrow{d^{(0)}} \mathbb{C}^1(A) \xrightarrow{d^{(1)}} \mathbb{C}^2(A) \xrightarrow{d^{(2)}} \mathbb{C}^3(A) \xrightarrow{d^{(3)}} \mathbb{C}^4(A) \xrightarrow{d^{(4)}} \dots,$$

where the space  $\mathbb{C}^p(A)$  of  $p$ -cochains is defined by  $\mathbb{C}^p(A) = 0$  for  $p \in -\mathbb{N}^*$ ,  $\mathbb{C}^0(A) = A$ , and for  $p \in \mathbb{N}^*$ ,  $\mathbb{C}^p(A)$  is the space of  $\mathbb{C}$ -linear maps from  $A^{\otimes p}$  to  $A$ . The differential  $d = \bigoplus_{i=0}^{\infty} d^{(p)}$  is given by

$$\begin{aligned} \forall C \in \mathbb{C}^p(A), \quad d^{(p)} C(a_0, \dots, a_p) &= a_0 C(a_1, \dots, a_p) \\ &\quad - \sum_{i=0}^{p-1} (-1)^i C(a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^{p-1} C(a_0, \dots, a_{p-1}) a_p. \end{aligned}$$

We may write it in terms of the Gerstenhaber bracket<sup>3</sup>  $[\cdot, \cdot]_G$  and of the product  $\mu$  of  $A$ , as follows:

$$d^{(p)} C = (-1)^{p+1} [\mu, C]_G.$$

<sup>2</sup>If  $A$  is a  $\mathbb{C}$ -algebra of finite type, then there exists some elements  $x_1, \dots, x_n \in A$ , algebraically independent on  $\mathbb{C}$ , such that  $A$  is integer on  $\mathbb{C}[x_1, \dots, x_n]$ .

<sup>3</sup>Recall that for  $F \in \mathbb{C}^p(A)$  and  $H \in \mathbb{C}^q(A)$ , the Gerstenhaber product is the element  $F \bullet H \in \mathbb{C}^{p+q-1}(A)$  defined by  $F \bullet H(a_1, \dots, a_{p+q-1}) = \sum_{i=0}^{p-1} (-1)^{i(q+1)} F(a_1, \dots, a_i, H(a_{i+1}, \dots, a_{i+q}), a_{i+q+1}, \dots, a_{p+q-1})$ , and the Gerstenhaber bracket is  $[F, H]_G := F \bullet H - (-1)^{(p-1)(q-1)} H \bullet F$ . See for example [5], and [2] p. 38.

Then we define the Hochschild cohomology of  $A$  as the cohomology of the Hochschild cohomological complex associated to  $A$ , i. e.  $HH^0(A) := \text{Ker } d^{(0)}$  and for  $p \in \mathbb{N}^*$ ,  $HH^p(A) := \text{Ker } d^{(p)} / \text{Im } d^{(p-1)}$ .

- Similarly, the Hochschild homological complex of  $A$  is

$$\dots \xrightarrow{d_5} C_4(A) \xrightarrow{d_4} C_3(A) \xrightarrow{d_3} C_2(A) \xrightarrow{d_2} C_1(A) \xrightarrow{d_1} C_0(A),$$

where the space of  $p$ -chains is given by  $C_p(A) = 0$  for  $p \in -\mathbb{N}^*$ ,  $C_0(A) = A$ , and for  $p \in \mathbb{N}^*$ ,  $C_p(A) = A \otimes A^{\otimes p}$ . The differential  $d = \bigoplus_{i=0}^{\infty} d_p$  is given by

$$\begin{aligned} d_p(a_0 \otimes a_1 \otimes \dots \otimes a_p) &= a_0 a_1 \otimes a_2 \otimes \dots \otimes a_p \\ &+ \sum_{i=1}^{p-1} (-1)^i a_0 \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_p + (-1)^p a_p a_0 \otimes a_1 \otimes \dots \otimes a_{p-1}. \end{aligned}$$

We define the Hochschild homology of  $A$  as the homology of the Hochschild homological complex associated to  $A$ , i. e.  $HH_0(A) := A / \text{Im } d_1$  and for  $p \in \mathbb{N}^*$ ,  $HH_p(A) := \text{Ker } d_p / \text{Im } d_{p+1}$ .

- We denote by  $\mathbb{C}[[\hbar]]$  (resp.  $A[[\hbar]]$ ) the algebra of formal power series in the parameter  $\hbar$ , with coefficients in  $\mathbb{C}$  (resp.  $A$ ). A deformation, or star-product, of the algebra  $A$  is a map  $*$  from  $A[[\hbar]] \times A[[\hbar]]$  to  $A[[\hbar]]$  which is  $\mathbb{C}[[\hbar]]$ -bilinear and such that

$$\begin{aligned} \forall k \in \mathbb{C}, \forall s \in A[[\hbar]], \quad k * s &= s * k = s, \\ \forall (s, t) \in A[[\hbar]]^2, \quad s * t &= st \mod \hbar A[[\hbar]], \\ \forall (s, t, u) \in A[[\hbar]]^3, \quad s * (t * u) &= (s * t) * u. \end{aligned}$$

This means that there exists a sequence of bilinear<sup>4</sup> maps  $C_j$  from  $A \times A$  to  $A$  of which the first term  $C_0$  is the multiplication of  $A$  and such that

$$\begin{aligned} \forall (a, b) \in A^2, \quad a * b &= \sum_{j=0}^{\infty} C_j(a, b) \hbar^j, \\ \forall n \in \mathbb{N}, \quad \sum_{i+j=n} C_i(a, C_j(b, c)) &= \sum_{i+j=n} C_i(C_j(a, b), c), \text{ i.e. } \sum_{i+j=n} [C_i, C_j]_G = 0. \end{aligned}$$

We say that the deformation is of order  $p$  if the previous formulae are satisfied (only) for  $n \leq p$ .

- The Hochschild cohomology plays an important role in the study of deformations of the algebra  $A$ , by helping us to classify them. In fact, if  $\pi \in C^2(A)$ , we may construct a first order deformation  $m$  of  $A$  such that  $m_1 = \pi$  if and only if  $\pi \in \text{Ker } d^{(2)}$ . Moreover, two first order deformations are equivalent<sup>5</sup> if and only if their difference is an element of  $\text{Im } d^{(1)}$ . So the set of equivalence classes of first order deformations is in bijection with  $HH^2(A)$ .

If  $m = \sum_{j=0}^p C_j \hbar^j$  (with  $C_j \in C^2(A)$ ) is a deformation of order  $p$ , then we may extend  $m$  to a deformation

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<sup>4</sup>In the definition of a star-product, we often assume that the bilinear maps  $C_j$  are bidifferential operators (see for example [2]). Here, we do not make this assumption.

<sup>5</sup>Two deformations  $m = \sum_{j=0}^p C_j \hbar^j$  (with  $C_j \in C^2(A)$ ) and  $m' = \sum_{j=0}^p C'_j \hbar^j$  (with  $C'_j \in C^2(A)$ ) of order  $p$  are called equivalent if there exists a sequence of linear maps  $\varphi_j$  from  $A$  to  $A$  of which the first term  $\varphi_0$  is the identity of  $A$  and such that

$$\begin{aligned} \forall a \in A, \quad \varphi(a) &= \sum_{j=0}^{\infty} \varphi_j(a) \hbar^j, \\ \forall n \in \mathbb{N}, \quad \sum_{i+j=n} \varphi_i(C_j(a, b)) &= \sum_{i+j+k=n} C'_i(\varphi_j(a), \varphi_k(b)). \end{aligned}$$

of order  $p + 1$  if and only if there exists  $C_{p+1} \in \mathcal{C}^2(A)$  such that

$$\forall (a, b, c) \in A^3, \sum_{i=1}^p (C_i(a, C_{p+1-i}(b, c)) - C_i(C_{p+1-i}(a, b), c)) = -d^{(2)} C_{p+1}(a, b, c),$$

$$\text{i.e. } \sum_{i=1}^p [C_i, C_{p+1-i}]_G = 2d^{(2)} C_{p+1}.$$

According to the graded Jacobi identity for  $[\cdot, \cdot]_G$ , the last sum belongs to  $\text{Ker } d^{(3)}$ . So  $HH^3(A)$  contains the obstructions to extend a deformation of order  $p$  to a deformation of order  $p + 1$ .

## 2.3 The Hochschild-Kostant-Rosenberg theorem

Let us now recall a fundamental result about the Hochschild cohomology of a smooth algebra. For more details, see the article [9] and the book [12], Section 3.4.

Let  $A$  be a  $\mathbb{C}$ -algebra. The  $A$ -module  $\Omega^1(A)$  of Kähler differentials is the  $A$ -module generated by the  $\mathbb{C}$ -linear symbols  $da$  for  $a \in A$ , such that  $d(ab) = a(bd) + b(da)$ .

For  $n \geq 2$ , the  $A$ -module of  $n$ -differential forms is the exterior product  $\Omega^n(A) := \Lambda^n \Omega^1(A)$ . By convention, we set  $\Omega^0(A) = A$ .

The antisymmetrization map  $\varepsilon_n$  is defined by

$$\varepsilon_n(a_1 \wedge \cdots \wedge a_n) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)},$$

where  $\varepsilon(\sigma)$  is the sign of the permutation  $\sigma$ .

Let us denote by  $HH_*(A) = \bigoplus_{n=0}^{\infty} HH_n(A)$  and  $\Omega^*(A)$  the exterior algebra of  $A$ .

### Theorem 6 (Hochschild-Kostant-Rosenberg)

Let  $A$  be a smooth algebra. Then the antisymmetrization map

$$\varepsilon_* : \Omega^*(A) \rightarrow HH_*(A)$$

is an isomorphism of graded algebras.

## 3 Hochschild cohomology of $\mathcal{X}_f$

### 3.1 Koszul complex

In the following, we shall use the results about the Koszul complex recalled below (see the appendix of [4]).

- We consider  $(f_1, \dots, f_m) \in \mathbb{C}[\mathbf{z}]^m$ , and we denote by  $A$  the quotient  $\mathbb{C}[\mathbf{z}] / \langle f_1, \dots, f_m \rangle$ . We assume that we have a *complete intersection*, i.e. the dimension of the set of solutions of the system

$$\{\mathbf{z} \in \mathbb{C}^n / f_1(\mathbf{z}) = \cdots = f_m(\mathbf{z}) = 0\}$$

is  $n - m$ .

- We consider the differential graded algebra

$$\Phi = A[\varepsilon_1, \dots, \varepsilon_n; u_1, \dots, u_m] = \frac{\mathbb{C}[z_1, \dots, z_n]}{\langle f_1, \dots, f_m \rangle}[\varepsilon_1, \dots, \varepsilon_n; u_1, \dots, u_m],$$

where  $\varepsilon_i := \frac{\partial}{\partial z_i}$  is an odd variable (i. e. the  $\varepsilon_i$ 's anticommute), and  $u_j$  an even variable (i. e. the  $u_j$ 's commute).

The algebra  $\Phi$  is endowed with the differential

$$\delta = \sum_{j=1}^n \sum_{i=1}^m \frac{\partial f_i}{\partial z_j} u_i \frac{\partial}{\partial \varepsilon_j},$$

and the Hodge grading, defined by  $\deg(z_i) = 0$ ,  $\deg(\varepsilon_i) = 1$ ,  $\deg(u_j) = 2$ .

The complex associated to  $\Phi$  is as follows:

$$\Phi(0) \xrightarrow{\tilde{0}} \Phi(1) \xrightarrow{\delta^{(1)}} \Phi(2) \xrightarrow{\delta^{(2)}} \Phi(3) \xrightarrow{\delta^{(3)}} \Phi(4) \xrightarrow{\delta^{(4)}} \dots$$

We may now state the main theorem which will allow us to calculate the Hochschild cohomology:

**Theorem 7** (*Kontsevich*)

*Under the previous assumptions, the Hochschild cohomology of  $A$  is isomorphic to the cohomology of the complex  $(\Phi, \delta)$  associated with the differential graded algebra  $\Phi$ .*

**Remark 8**

*Theorem 7 may be seen as a generalization of the Hochschild-Kostant-Rosenberg theorem to the case of non-smooth spaces.*

Set  $H^0 := A$ ,  $H^1 := \text{Ker } \delta^{(1)}$  and for  $j \geq 2$ ,  $H^j := \text{Ker } \delta^{(j)} / \text{Im } \delta^{(j-1)}$ .

According to Theorem 7, we have, for  $p \in \mathbb{N}$ ,  $HH^p(A) \simeq H^p$ .

### 3.2 Cohomology spaces

• We consider now the case  $A := \mathbb{C}[z_1, z_2, z_3] / \langle f \rangle$  and we want to calculate its Hochschild cohomology. The different spaces of the complex are now given by

$$\begin{aligned} \Phi(0) &= A, \quad \Phi(1) = A\varepsilon_1 \oplus A\varepsilon_2 \oplus A\varepsilon_3 \\ \forall p \in \mathbb{N}^*, \quad \Phi(2p) &= Au_1^p \oplus Au_1^{p-1}\varepsilon_1\varepsilon_2 \oplus Au_1^{p-1}\varepsilon_2\varepsilon_3 \oplus Au_1^{p-1}\varepsilon_3\varepsilon_1, \\ \forall p \in \mathbb{N}^*, \quad \Phi(2p+1) &= Au_1^p\varepsilon_1 \oplus Au_1^p\varepsilon_2 \oplus Au_1^p\varepsilon_3 \oplus Au_1^{p-1}\varepsilon_1\varepsilon_2\varepsilon_3. \end{aligned}$$

This defines the bases  $\mathcal{B}_p$ . We have  $\frac{\partial}{\partial \varepsilon_1}(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3) = 1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \varepsilon_2 \wedge \varepsilon_3 \wedge 1$ , thus  $\delta^{(3)}(\varepsilon_1\varepsilon_2\varepsilon_3) = \frac{\partial f}{\partial z_1}u_1\varepsilon_2\varepsilon_3 + \frac{\partial f}{\partial z_2}u_1\varepsilon_3\varepsilon_1 + \frac{\partial f}{\partial z_3}u_1\varepsilon_1\varepsilon_2$ .

We set  $Df := \begin{pmatrix} \partial_3 f & \partial_1 f & \partial_2 f \end{pmatrix}$ . The matrices  $[\delta^{(j)}]$  of  $\delta^{(j)}$  in the bases  $\mathcal{B}_j$ ,  $\mathcal{B}_{j+1}$  are therefore given by

$$[\delta^{(1)}] = \begin{pmatrix} {}^t\nabla f \\ \mathbf{0}_{3,3} \end{pmatrix}, \quad \forall p \in \mathbb{N}^*, \quad [\delta^{(2p)}] = \begin{pmatrix} 0 & \partial_2 f & 0 & -\partial_3 f \\ 0 & -\partial_1 f & \partial_3 f & 0 \\ 0 & 0 & -\partial_2 f & \partial_1 f \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [\delta^{(2p+1)}] = \begin{pmatrix} {}^t\nabla f & 0 \\ \mathbf{0}_{3,3} & {}^t Df \end{pmatrix}.$$

• We deduce

$$H^0 = A.$$

$$\begin{aligned}
H^1 &= \{g_1 \varepsilon_1 + g_2 \varepsilon_2 + g_3 \varepsilon_3 \mid (g_1, g_2, g_3) \in A^3 \text{ and } g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f = 0\} \\
&\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \mid \mathbf{g} \cdot \nabla f = 0 \right\}. \\
H^2 &= \frac{\{g_0 u_1 + g_3 \varepsilon_1 \varepsilon_2 + g_1 \varepsilon_2 \varepsilon_3 + g_2 \varepsilon_3 \varepsilon_1 \mid (g_0, g_1, g_2, g_3) \in A^4 \text{ and } g_3 \partial_2 f - g_2 \partial_3 f = g_1 \partial_3 f - g_3 \partial_1 f = g_2 \partial_1 f - g_1 \partial_2 f = 0\}}{\{(g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f) u_1 \mid (g_1, g_2, g_3) \in A^3\}} \\
&\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \mid \nabla f \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\} \Bigg/ \left\{ \begin{pmatrix} \mathbf{g} \cdot \nabla f \\ \mathbf{0}_{3,1} \end{pmatrix} \mid \mathbf{g} \in A^3 \right\} \\
&\simeq \frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle_A} \oplus \{\mathbf{g} \in A^3 \mid \nabla f \wedge \mathbf{g} = 0\}.
\end{aligned}$$

For  $p \geq 2$ ,

$$\begin{aligned}
H^{2p} &= \frac{\left\{ g_0 u_1^p + g_3 u_1^{p-1} \varepsilon_1 \varepsilon_2 + g_1 u_1^{p-1} \varepsilon_2 \varepsilon_3 + g_2 u_1^{p-1} \varepsilon_3 \varepsilon_1 \mid (g_0, g_1, g_2, g_3) \in A^4 \text{ and } g_3 \partial_2 f - g_2 \partial_3 f = g_1 \partial_3 f - g_3 \partial_1 f = g_2 \partial_1 f - g_1 \partial_2 f = 0 \right\}}{\{(g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f) u_1^p + g_0 (\partial_3 f u_1^{p-1} \varepsilon_1 \varepsilon_2 + \partial_1 f u_1^{p-1} \varepsilon_2 \varepsilon_3 + \partial_2 f u_1^{p-1} \varepsilon_3 \varepsilon_1) \mid (g_0, g_1, g_2, g_3) \in A^3\}} \\
&\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \mid \nabla f \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\} \Bigg/ \left\{ \begin{pmatrix} \mathbf{g} \cdot \nabla f \\ g_0 \partial_1 f \\ g_0 \partial_2 f \\ g_0 \partial_3 f \end{pmatrix} \mid \mathbf{g} \in A^3 \text{ and } g_0 \in A \right\} \\
&\simeq \frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle_A} \oplus \frac{\{\mathbf{g} \in A^3 \mid \nabla f \wedge \mathbf{g} = 0\}}{\{g \nabla f \mid g \in A\}}.
\end{aligned}$$

For  $p \in \mathbb{N}^*$ ,

$$\begin{aligned}
H^{2p+1} &= \frac{\left\{ g_1 u_1^p \varepsilon_1 + g_2 u_1^p \varepsilon_2 + g_3 u_1^p \varepsilon_3 + g_0 u_1^{p-1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \mid (g_0, g_1, g_2, g_3) \in A^4 \text{ and } g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f = 0 \right. \\
&\quad \left. g_0 \partial_3 f = g_0 \partial_1 f = g_0 \partial_2 f = 0 \right\}}{\{(g_3 \partial_2 f - g_2 \partial_3 f) u_1^p \varepsilon_1 + (g_1 \partial_3 f - g_3 \partial_1 f) u_1^p \varepsilon_2 + (g_2 \partial_1 f - g_1 \partial_2 f) u_1^p \varepsilon_3 \mid (g_1, g_2, g_3) \in A^3\}} \\
&\simeq \left\{ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_0 \end{pmatrix} \in A^4 \mid \nabla f \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right. \\
&\quad \left. g_0 \partial_3 f = g_0 \partial_1 f = g_0 \partial_2 f = 0 \right\} \Bigg/ \left\{ \begin{pmatrix} \nabla f \wedge \mathbf{g} \\ 0 \end{pmatrix} \mid \mathbf{g} \in A^3 \right\} \\
&\simeq \frac{\{\mathbf{g} \in A^3 \mid \nabla f \cdot \mathbf{g} = 0\}}{\{\nabla f \wedge \mathbf{g} \mid \mathbf{g} \in A^3\}} \oplus \{g \in A \mid g \partial_3 f = g \partial_1 f = g \partial_2 f = 0\}.
\end{aligned}$$

The following section will allow us to make those various spaces more explicit.

### 3.3 Hochschild cohomology of $\mathcal{X}_f$

The three partial derivative of  $f$  are  $\partial_1 f = 3z_1^2 + 3qz_2z_3$ ,  $\partial_2 f = 3z_2^2 + 3qz_1z_3$ , and  $\partial_3 f = 3z_3^2 + 3qz_1z_2$ . So, according to Euler's formula, we get  $z_1 \partial_1 f + z_2 \partial_2 f + z_3 \partial_3 f = 3f$ , hence the inclusion  $\langle f \rangle \subset \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle$ . We immediately deduce the isomorphism:

$$\frac{A}{\langle \nabla f \rangle_A} \simeq \frac{\mathbb{C}[\mathbf{z}]}{\langle \nabla f \rangle}.$$

The following lemma gives an explicit expression of this quotient.

#### Lemma 9

The vector space  $\mathbb{C}[\mathbf{z}] / \langle \nabla f \rangle$  is isomorphic to

$$\text{Vect}(1, z_3, z_3^2, z_3^3, z_2, z_2 z_3, z_2^2, z_1).$$



Proof:

We use Maple: a Groebner basis of the ideal  $\langle \nabla f \rangle$  is

$$[z_3^4, z_3^2 z_2, z_2^2 z_3, -z_3^3 + z_3^2, z_2^2 + q z_1 z_3, z_3^2 + q z_1 z_2, z_1^2 + q z_2 z_3].$$

So the lemma results from Macaulay's theorem. ■

**Lemma 10**

The set of solutions of the equation  $p\partial_1 f = 0$  in the quotient algebra  $\mathbb{C}[\mathbf{z}] / \langle f, \partial_2 f, \partial_3 f \rangle$  is the vector space

$$\text{Vect}(z_1, z_1^2, z_2^2, z_2^2 z_3, z_3^2, z_3^2 z_2, z_2^2 z_3^2, z_3^3).$$

Proof:

We use Maple and the multivariate division. A Groebner basis of the ideal  $\langle f, \partial_2 f, \partial_3 f \rangle$  is

$$[z_3^4, z_3^3 z_2, -z_3^3 + z_3^2, z_2^2 + q z_1 z_3, z_3^2 + q z_1 z_2, -z_3^3 + z_1^3].$$

So, according to Macaulay's theorem, a basis of the quotient space is

$$(1, z_1, z_1^2, z_2, z_2^2, z_3, z_2 z_3, z_2^2 z_3, z_3^2, z_3^2 z_2, z_2^2 z_3^2, z_3^3).$$

Let  $p := a_1 + a_2 z_1 + a_3 z_1^2 + a_4 z_2 + a_5 z_2^2 + a_6 z_3 + a_7 z_2 z_3 + a_8 z_2^2 z_3 + a_9 z_3^2 + a_{10} z_3^2 z_2 + a_{11} z_2^2 z_3^2 + a_{12} z_3^3$  an element of the quotient space. Thanks to the multivariate division by the Groebner basis, the normal form of  $p\partial_1 f$  with respect to the ideal  $\langle f, \partial_2 f, \partial_3 f \rangle$  is

$$3 a_1 z_1^2 + 3 q a_7 z_2^2 z_3^2 + 3 \frac{a_7}{q^2} z_2^2 z_3^2 + 3 q a_4 z_2^2 z_3 + 3 \frac{a_4}{q^2} z_3 z_2^2 + 3 q a_6 z_3^2 z_2 + 3 \frac{a_6}{q^2} z_2 z_3^2 + 3 q a_1 z_2 z_3.$$

So,  $p\partial_1 f$  belongs to the ideal  $\langle f, \partial_2 f, \partial_3 f \rangle$  if and only if  $a_1 = a_4 = a_6 = a_7 = 0$ . Hence the proof of the lemma. ■

We are now able to solve the equation  $\mathbf{g} \cdot \nabla f = 0$  in  $A^3$ . It is the object of the following lemma:

**Lemma 11**

The set of solutions of the equation  $\mathbf{g} \cdot \nabla f = 0$  in  $A^3$  is isomorphic to the set

$$\{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \oplus \mathbb{C}^8.$$

Proof:

We have solve the equation  $g \cdot \nabla f = 0$  in  $A$ , i.e.

$$g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f = 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f \rangle. \quad (3)$$

Let  $g \in \mathbb{C}[\mathbf{z}]$  be a solution of  $g \cdot \nabla f = 0$ , i.e.  $g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f \in \langle f \rangle$ .

Hence  $g_1 \partial_1 f \in \langle f, \partial_2 f, \partial_3 f \rangle$ . So, according to Lemma 10, we have

$$g_1 = \varepsilon f + \eta \partial_2 f + \zeta \partial_3 f + \varphi, \quad (4)$$

where

$$\varphi = \sum_{i=1}^2 a_i z_1^i + \sum_{i=0}^2 b_i z_2^2 z_3^i + \sum_{i=0}^1 c_i z_3^2 z_2^i + d z_3^3,$$

with  $\varepsilon, \eta, \zeta \in \mathbb{C}[\mathbf{z}]$  and  $a_i, b_i, c_i, d \in \mathbb{C}$ . Now, we have the Euler formula:

$$z_1 \partial_1 f + z_2 \partial_2 f + z_3 \partial_3 f = 3f. \quad (5)$$

Moreover, we have the two following key-relations:

$$z_2^2 \partial_1 f = (z_1^2 + 2q z_2 z_3) \partial_2 f - 3q z_3 f + q z_3^2 \partial_3 f, \quad (6)$$

$$z_3^2 \partial_1 f = (z_1^2 + 2qz_2z_3) \partial_3 f - 3qz_2f + qz_2^2 \partial_2 f, \quad (7)$$

Thus

$$\begin{aligned} g_1 \partial_1 f &= \left( \sum_{i=1}^2 a_i z_1^{i-1} \right) z_1 \partial_1 f + \left( \sum_{i=0}^2 b_i z_3^i \right) z_3^2 \partial_1 f \\ &\quad + \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) z_3^2 \partial_1 f + \varepsilon f \partial_1 f + \eta \partial_1 f \partial_2 f + \zeta \partial_1 f \partial_3 f \\ &= \left( \sum_{i=1}^2 a_i z_1^{i-1} \right) (3f - z_2 \partial_2 f - z_3 \partial_3 f) + \left( \sum_{i=0}^2 b_i z_3^i \right) ((z_1^2 + 2qz_2z_3) \partial_2 f - 3qz_3f + qz_3^2 \partial_3 f) \\ &\quad + \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) ((z_1^2 + 2qz_2z_3) \partial_3 f - 3qz_2f + qz_2^2 \partial_2 f) + \varepsilon f \partial_1 f + \eta \partial_1 f \partial_2 f + \zeta \partial_1 f \partial_3 f. \end{aligned}$$

Then, Equation (3) becomes

$$\begin{aligned} &\left( - \sum_{i=1}^2 a_i z_1^{i-1} z_2 + (z_1^2 + 2qz_2z_3) \sum_{i=0}^2 b_i z_3^i + qz_2^2 \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + g_2 + \eta \partial_1 f \right) \partial_2 f \\ &\quad - \left( - \sum_{i=1}^2 a_i z_1^{i-1} z_3 + qz_3^2 \sum_{i=0}^2 b_i z_3^i + (z_1^2 + 2qz_2z_3) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + g_3 + \zeta \partial_1 f \right) \partial_3 f \\ &= 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f \rangle. \end{aligned}$$

Hence

$$\left( - \sum_{i=1}^2 a_i z_1^{i-1} z_2 + (z_1^2 + 2qz_2z_3) \sum_{i=0}^2 b_i z_3^i + qz_2^2 \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + g_2 + \eta \partial_1 f \right) \partial_2 f = 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f, \partial_3 f \rangle.$$

Now, as  $\langle f, \partial_3 f \rangle$ ,  $\partial_2 f$  is a regular sequence, we have  $\text{Ann}_{\langle f, \partial_3 f \rangle}(\partial_2 f) = \langle f, \partial_3 f \rangle : \langle \partial_2 f \rangle = \langle f, \partial_3 f \rangle$ , therefore

$$- \sum_{i=1}^2 a_i z_1^{i-1} z_2 + (z_1^2 + 2qz_2z_3) \sum_{i=0}^2 b_i z_3^i + qz_2^2 \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + g_2 + \eta \partial_1 f \in \mathbb{C}[\mathbf{z}] / \langle f, \partial_3 f \rangle,$$

i.e.

$$g_2 = \alpha f + \beta \partial_3 f - \eta \partial_1 f + \psi \quad (8)$$

with

$$\psi := \sum_{i=1}^2 a_i z_1^{i-1} z_2 - (z_1^2 + 2qz_2z_3) \sum_{i=0}^2 b_i z_3^i - qz_2^2 \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right).$$

By substituting, we get

$$\varphi \partial_1 f + \psi \partial_2 f + (\alpha \partial_2 f + \varepsilon \partial_1 f) f + (\beta \partial_2 f + \zeta \partial_1 f + g_3) \partial_3 f = 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f \rangle. \quad (9)$$

Now, according to (5), (6), (7), we have

$$\begin{aligned} \varphi \partial_1 f + \psi \partial_2 f &= (z_2^2 \partial_1 f - (z_1^2 + 2qz_2z_3) \partial_2 f) \sum_{i=0}^2 b_i z_3^i + (z_3^2 \partial_1 f - qz_2^2 \partial_2 f) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) \\ &\quad + (z_1 \partial_1 f + z_2 \partial_2 f) \sum_{i=1}^2 a_i z_1^{i-1} \\ &= (qz_3^2 \partial_3 f - 3qz_3f) \sum_{i=0}^2 b_i z_3^i + ((z_1^2 + 2qz_2z_3) \partial_3 f - 3qz_2f) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) \\ &\quad + (3f - z_3 \partial_3 f) \sum_{i=1}^2 a_i z_1^{i-1}, \end{aligned}$$

therefore

$$\begin{aligned} \varphi\partial_1 f + \psi\partial_2 f &= \left( qz_3^2 \sum_{i=0}^2 b_i z_3^i + (z_1^2 + 2qz_2 z_3) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) - z \sum_3 \sum_{i=1}^2 a_i z_1^{i-1} \right) \partial_3 f \\ &\quad + \left( -3qz_3 \sum_{i=0}^2 b_i z_3^i - 3qz_2 \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + 3 \sum_{i=1}^2 a_i z_1^{i-1} \right) f. \end{aligned} \quad (10)$$

By another substituting, we obtain

$$\left( qz_3^2 \sum_{i=0}^2 b_i z_3^i + (z_1^2 + 2qz_2 z_3) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) - z_3 \sum_{i=1}^2 a_i z_1^{i-1} + \zeta\partial_1 f + \beta\partial_2 f + g_3 \right) \partial_3 f = 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f \rangle.$$

As  $\partial_3 f$  and  $f$  are coprime, we deduce

$$g_3 = \gamma f - \zeta\partial_1 f - \beta\partial_2 f + \chi, \quad (11)$$

with  $\gamma \in \mathbb{C}[\mathbf{z}]$ , where

$$\chi := -qz_3^2 \sum_{i=0}^2 b_i z_3^i - (z_1^2 + 2qz_2 z_3) \left( dz_3 + \sum_{i=0}^1 c_i z_2^i \right) + z_3 \sum_{i=1}^2 a_i z_1^{i-1}.$$

Now, let  $\mathbf{g}$  be defined by the Equations (4), (8), (11). Then

$$\mathbf{g} \cdot \nabla f = \varphi\partial_1 f + \psi\partial_2 f + \chi\partial_3 f + (\varepsilon\partial_1 f + \alpha\partial_2 f + \gamma\partial_3 f)f \in \langle f \rangle,$$

according to Relation (10).

Finally, we have proved

$$\{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\} = \left\{ \nabla f \wedge \begin{pmatrix} \beta \\ -\zeta \\ \eta \end{pmatrix} + \begin{pmatrix} \varphi \\ \psi \\ \chi \end{pmatrix} \right\},$$

hence

$$\{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\} \simeq \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \oplus \mathbb{C}^8. \blacksquare$$

So, the cohomology spaces of odd degrees are given by

$$\forall p \in \mathbb{N}^*, \quad \begin{aligned} H^1 &\simeq \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \oplus \mathbb{C}^8, \\ H^{2p+1} &\simeq \mathbb{C}^8. \end{aligned}$$

We now determine the set  $\{\mathbf{g} \in A^3 / \nabla f \wedge \mathbf{g} = 0\}$ :

**Lemma 12**

The set of solutions of the equation  $\nabla f \wedge \mathbf{g} = 0$  in  $A$  is isomorphic to the set  $\{g\nabla f / g \in A\}$ .

Proof:

Let  $\mathbf{g} \in A^3$  be such that  $\nabla f \wedge \mathbf{g} = 0$ , i.e.

$$g_3\partial_2 f - g_2\partial_3 f = g_1\partial_3 f - g_3\partial_1 f = g_2\partial_1 f - g_1\partial_2 f = 0 \mod \langle f \rangle.$$

Then

$$g_3\partial_2 f - g_2\partial_3 f = 0 \mod \langle f \rangle, \quad (12)$$

hence  $g_3\partial_2 f = 0 \mod \langle f, \partial_3 f \rangle$ .

As  $\langle f, \partial_3 f \rangle, \partial_2 f$  is a regular sequence, we have  $\langle f, \partial_3 f \rangle : \langle \partial_2 f \rangle = \langle f, \partial_3 f \rangle$ , thus  $g_3 \in \langle f, \partial_3 f \rangle$ , i.e.

$$g_3 = \alpha f + \beta \partial_3 f.$$

We get

$$(\alpha f + \beta \partial_3 f) \partial_2 f - g_2 \partial_3 f = 0 \text{ in } \mathbb{C}[\mathbf{z}] / \langle f \rangle,$$

hence  $(\beta \partial_2 f - g_2) \partial_3 f = 0$  in  $\mathbb{C}[\mathbf{z}] / \langle f \rangle$ . As  $f$  and  $\partial_3 f$  are coprime, we deduce  $\beta \partial_2 f - g_2 \in \langle f \rangle$ , i.e.  $g_2 = \beta \partial_2 f - \gamma f$ .

By substituting in  $g_1 \partial_3 f - g_3 \partial_1 f = 0 \pmod{\langle f \rangle}$ , we obtain  $(g_1 - \beta \partial_1 f) \partial_3 f = 0 \pmod{\langle f \rangle}$ .

Similarly, we have  $g_1 - \beta \partial_1 f \in \langle f \rangle$ , i.e.  $g_1 = \beta \partial_1 f + \varepsilon f$ .

Finally,

$$\{\mathbf{g} \in A^3 / \nabla f \wedge \mathbf{g} = 0\} = \{g \nabla f / g \in A\}. \blacksquare$$

So, the cohomology spaces of even degrees are

$$\begin{aligned} H^2 &\simeq \{g \nabla f / g \in A\} \oplus \mathbb{C}^8 \\ \forall p \geq 2, H^{2p} &\simeq \mathbb{C}^8 \end{aligned}$$

Finally, the Hochschild cohomology is given by the following theorem.

**Theorem 13**

*Let  $HH^p$  be the space of Hochschild cohomology of degree  $p$ . Then we have*

$$HH^0 = A, HH^1 = \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \oplus \mathbb{C}^8, HH^2 = \{g \nabla f / g \in A\} \oplus \mathbb{C}^8, \text{ and } \forall p \geq 3, HH^p = \mathbb{C}^8.$$

## 4 Hochschild homology of $\mathcal{X}_f$

### 4.1 Koszul complex

- There is an analogous of Theorem 7 for the Hochschild homology: we consider the complex

$$\Psi = A[\zeta_1, \dots, \zeta_n; v_1, \dots, v_m],$$

where  $\zeta_i$  is an odd variable and  $v_j$  an even variable.  $\Psi$  is endowed with the differential

$$\vartheta = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial f_j}{\partial z_i} \zeta_i \frac{\partial}{\partial v_j},$$

and the Hodge grading, defined by  $\deg(z_i) = 0$ ,  $\deg(\zeta_i) = -1$ ,  $\deg(v_j) = -2$ .

$$\dots \xrightarrow{\vartheta^{(-5)}} \Psi(-4) \xrightarrow{\vartheta^{(-4)}} \Psi(-3) \xrightarrow{\vartheta^{(-3)}} \Psi(-2) \xrightarrow{\vartheta^{(-2)}} \Psi(-1) \xrightarrow{\vartheta^{(-1)}} \Psi(0)$$

**Theorem 14** (*Kontsevich*)

*Under the previous assumptions, the Hochschild homology of  $A$  is isomorphic to the cohomology of the complex  $(\Psi, \vartheta)$ .*

Set  $L^0 := A / \text{Im } \vartheta^{(-1)}$ , and for  $p \geq 1$ ,  $L^{-p} := \text{Ker } \vartheta^{(-p)} / \text{Im } \vartheta^{(-p-1)}$ .

According to Theorem 14, we have, for  $p \in \mathbb{N}$ ,  $HH_p(A) \simeq L^{-p}$ .

## 4.2 Hochschild homology of $\mathcal{X}_f$

- Here, we have

$$\begin{aligned}\Psi(0) &= A, \\ \Psi(-1) &= A\zeta_1 \oplus A\zeta_2 \oplus A\zeta_3, \\ \forall p \in \mathbb{N}^*, \quad \Psi(-2p) &= Av_1^p \oplus Av_1^{p-1}\zeta_1\zeta_2 \oplus Av_1^{p-1}\zeta_2\zeta_3 \oplus Av_1^{p-1}\zeta_3\zeta_1, \\ \forall p \in \mathbb{N}^*, \quad \Psi(-2p-1) &= Av_1^p\zeta_1 \oplus Av_1^p\zeta_2 \oplus Av_1^p\zeta_3 \oplus Av_1^{p-1}\zeta_1\zeta_2\zeta_3.\end{aligned}$$

This defines the bases  $\mathcal{V}_p$ . The differential is  $\vartheta = (\zeta_1\partial_1f + \zeta_2\partial_2f + \zeta_3\partial_3f)\frac{\partial}{\partial v_1}$ .

By setting  $Df := \begin{pmatrix} \partial_3f & \partial_1f & \partial_2f \end{pmatrix}$ , we deduce the matrices  $[\vartheta^{(-j)}]$  of  $\vartheta^{(-j)}$  in the bases  $\mathcal{V}_{-j}$ ,  $\mathcal{V}_{-j+1}$ :

$$\begin{aligned}[\vartheta^{(-2)}] &= \begin{pmatrix} \nabla f & \mathbf{0}_{3,3} \end{pmatrix}, \\ \forall p \geq 2, [\vartheta^{(-2p)}] &= \begin{pmatrix} \nabla f & \mathbf{0}_{3,3} \\ 0 & (p-1)Df \end{pmatrix}, \quad \forall p \geq 1, [\vartheta^{(-2p-1)}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -p\partial_2f & p\partial_1f & 0 & 0 \\ 0 & -p\partial_3f & p\partial_2f & 0 \\ p\partial_3f & 0 & -p\partial_1f & 0 \end{pmatrix}.\end{aligned}$$

- The cohomology spaces read :

$$L^0 = A,$$

$$L^{-1} = \frac{A^3}{\{g\nabla f / g \in A\}},$$

$$L^{-2} = \{g \in A / g\partial_1f = g\partial_2f = g\partial_3f = 0\} \oplus \frac{A^3}{\{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\}}.$$

For  $p \geq 2$ ,

$$L^{-2p} \simeq \{g \in A / g\partial_1f = g\partial_2f = g\partial_3f = 0\} \oplus \frac{\{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\}}{\{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\}}.$$

For  $p \in \mathbb{N}^*$ ,

$$L^{-2p-1} \simeq \frac{\{\mathbf{g} \in A^3 / \nabla f \wedge \mathbf{g} = 0\}}{\{g\nabla f / g \in A\}} \oplus \frac{A}{\langle \nabla f \rangle_A}.$$

- We have  $\{g \in A / g\partial_1f = g\partial_2f = g\partial_3f = 0\} = \{0\}$ , and according to Euler's formula,  $\frac{A}{\langle \nabla f \rangle_A} \simeq \frac{\mathbb{C}[\mathbf{z}]}{\langle \nabla f \rangle}$ . With the exception of  $A^3 / \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\}$ , the spaces quoted above have been computed in the preceding section, in particular  $A^3 / A\nabla f \simeq \nabla f \wedge A^3$  (See Remark ). Now  $\{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\} \subset \{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\}$ , therefore

$$\dim(A^3 / \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\}) \geq \dim(A^3 / \{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\}),$$

and  $A^3 / \{\mathbf{g} \in A^3 / \mathbf{g} \cdot \nabla f = 0\} \simeq \{\mathbf{g} \cdot \nabla f / \mathbf{g} \in A^3\}$ . As the map

$$g \in A \mapsto \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix} \cdot \nabla f \in \{\mathbf{g} \cdot \nabla f / \mathbf{g} \in A^3\}$$

is injective,  $A^3 / \{\nabla f \wedge \mathbf{g} / \mathbf{g} \in A^3\}$  is infinite dimensional.

- Finally, the Hochschild homology is given by the following theorem.

### Theorem 15

Let  $HH_p$  be the space of Hochschild homology of degree  $p$ . Then we have

$$HH_0 = A, \quad HH_1 = \nabla f \wedge A^3, \quad HH_2 = A^3 / (\nabla f \wedge A^3), \quad \text{and } \forall p \geq 3, \quad HH_p = \mathbb{C}^8.$$

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